Int. J. Solids Structures Vol. 30, No. 17, pp. 2287–2299, 1993
Printed in Great Britain

0020–7683/93 $6.00+ .00
Pergamon Press Ltd

NONLINEAR DYNAMICS OF A FLEXIBLE BEAM IN A CENTRAL GRAVITATIONAL FIELD—I. EQUATIONS OF MOTION

MARCELO R. M. CRESPO DA SILVA and CLIFFORD L. ZARETZKY
Department of Mechanical Engineering, Aeronautical Engineering and Mechanics,
Rensselaer Polytechnic Institute, Troy, NY 12180-3590, U.S.A.

(*Received* 7 *December* 1992; *in revised form* 19 *March* 1993)

Abstract—The complete nonlinear differential equations governing the nonlinear motions of a beam able to undergo bending and pitching in space, are formulated in this paper. The formulation is based on a variational principle and accounts for all the nonlinearities due to deformation and gravity gradient effects. The nonlinearities due to deformation arise due to geometric effects, which consist of nonlinear curvature and nonlinear inertia terms. Expanded equations governing the nonlinear perturbed motion about an equilibrium are also developed for the case when the beam is in circular orbit. Such equations are suited for a perturbation analysis of the motion, and nonlinearities up to cubic order in a bookkeeping parameter are retained in them. Nonlinear motions involving interactions between bending and pitching of the beam are investigated in Part II of this work using the equations developed here.

INTRODUCTION

Some engineering structures are constructed to operate in the space environment where their behavior is affected by gravity gradient moments, and by external forces such as due to aerodynamics (for low earth orbits) and solar radiation pressure, control forces, etc. Consideration of flexibility of such structures is of utmost importance to the understanding of their dynamic behavior [see, for example, Etkin and Hughes (1967), Krishna and Bainum (1984), Modi (1974) and Modi and Ng (1989)]. Many components of such structures are long members that can be modeled as beams. For an orbiting beam, gravity gradient effects cause it to oscillate relative to an orbital reference frame with pitch frequencies that are of the order of the orbital angular speed.

Under some conditions, linear mathematical models do not predict the actual motion of a space structure since such models do not disclose the various types of dynamic phenomena that are caused by the nonlinearities in the equations of motion. For a beam, for example, even if its material is linear, the differential equations of motion for the system contain a number of nonlinearities due to deformation. Such nonlinearities are due to the deformation of the structure and to the gravity gradient effect (which depends on the deformation and on the orientation of the beam in space). They include inertia terms, and terms that arise from the expression for the curvature of the beam, which involve nonlinear terms in the elastic deformations. It turns out that nonlinearities that are present in the expression for the curvature are of the same order of inertia nonlinearities, and of nonlinearities due to the gravity gradient effect. Thus, care should be taken so that all such nonlinearities are retained in the formulation in a consistent manner.

A number of studies undertaken in the past were restricted to linearized models to describe the flexibility of both free–free beam-like structures and flexible appendages attached to rigid satellites. Few analytical studies have dealt with the nonlinear dynamics of flexible structures in orbit modeled as beams. Of these, the pioneering works of Ashley (1967), Kumar and Bainum (1980), Bainum and Kumar (1982), and Budynas and Poli (1971), are closely related to the present work. Those works considered the dynamics of a slender beam in orbit undergoing small pitch and flexure motions.

In the work presented in the references mentioned above, the dynamics of a free–free beam in orbit was addressed. Kumar and Bainum (1980) and Bainum and Kumar (1982) addressed the planar motion of a long slender beam in a circular orbit undergoing pitch and flexural motions, and the motion and stability of space beams about a nominal local horizontal orientation. As in Ashley (1967), small motions were assumed and it was concluded that the pitch motion is essentially decoupled from the elastic motion and that the elastic motion is essentially governed by Hill's equation with the pitching motion acting as a parametric excitation to the elastic bending motion. Nonlinearities such as those due to nonlinear terms in the expression for the beam's curvature were not considered. These and other nonlinearities that are accounted for in this paper were not considered in the work of Budynas and Poli (1971) either, where the planar motion of a large flexible satellite consisting of a compact rigid body containing two flexible antennae located 180° from one another was studied. In the present work, which is divided into two parts, it is shown that other nonlinearities due to the deformation of the structure can play a dominant role in the motion of the structure under certain conditions of practical interest.

The formulation of the nonlinear differential equations of motion for a beam undergoing pitching and bending motions in the plane of the motion of the center of mass of the structure in space is addressed in this paper. The formulation presented here is based on that developed by Crespo da Silva and Glynn (1978a, b) and by Crespo da Silva (1991), and accounts for all nonlinear terms that arise from geometric effects. It also accounts for all the nonlinearities arising from gravity gradient effects acting along the beam, and from coupling between the elastic and the pitch motions of the beam. The formulation is based on a variational approach, which also yields a general boundary condition equation that involves the deformation variables for the motion. In order to be able to investigate the motion by analytical techniques, the full nonlinear differential equations of motion developed here are also expanded about an equilibrium solution into polynomial nonlinearities up to cubic order in an arbitrary "bookkeeping parameter". The expanded equations are especially suited for a perturbation analysis of the motion of the beam.

MATHEMATICAL MODEL AND BASIC ASSUMPTIONS

In this paper, the nonlinear differential equations that govern the motion of a beam in the plane of the motion of its center of mass are formulated. The equations developed here are applicable to the case where the beam is subjected to the force of a central attracting point, E, that represents the center of mass of a spherical planet of radius R_E and very large mass m_E. The small, long term effect of the motion of E around another central attracting body is neglected, and E is treated as if it were an inertial point. The beam may be subjected to small distributed and/or concentrated forces. These forces are assumed to be small enough so that their effect on the motion of its center of mass due to the inverse-square law gravitational attraction of E is neglected.

Consider a free–free thin beam of length $L \ll R_E$, specific mass m kg m^{-1}, made of Hookean material, and subjected to the gravitational attraction of E. As shown in Fig. 1, the center of mass of the beam, C, moves in a trajectory in space due to the central attraction of E. The location of C relative to E is described by the distance $R_c(t)$ and by the angle $\phi(t)$, where t denotes time. The line EC rotates in space with an angular velocity equal to $\dot{\phi}$ rads s^{-1}. The beam is assumed to be straight when it is undeformed. Figure 1 shows the

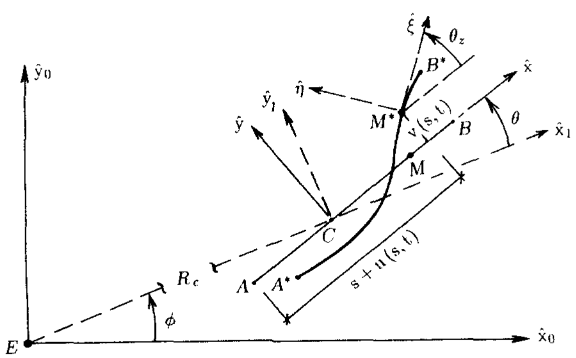

Fig. 1. Free–free beam subjected to the gravitational force of a central attracting body.

reference line of the beam in both its undeformed (AB) and deformed (A^*B^*) configurations. The reference line AB is chosen to be a principal axis of the deformed beam. Due to the gravity gradient effect, that axis tends to point toward the center of attraction E (e.g. Hughes, 1986). The axes (x_0, y_0) shown in that figure are inertial, while the axes (η, ζ) are taken to be the principal axes of the beam's cross-section, normal to the ξ axis, at position s. Here s denotes the arc-length along the deformed beam. Carets (^) in that figure are used to indicate unit vectors along the different axes. When the beam is in its undeformed state, the unit vectors $\hat{\xi}$ and $\hat{\eta}$ are aligned with $\hat{x}$ and $\hat{y}$, respectively; $\hat{x}$ is oriented along line AB, which is pitched by an angle θ with respect to the orbital reference unit vector $\hat{x}_1$ in the direction from E to C.

During deformation, an arbitrary point M on the beam's reference line moves to position M^*, thus undergoing an elastic displacement $u(s,t)\hat{x}+v(s,t)\hat{y}$. The position $s = 0$ corresponds to point A^* in Fig. 1, while $s = L$ corresponds to point B^*. With $x_c \triangleq |\overrightarrow{AC}|$, and since point C is the center of mass of the beam, it follows that

$$\int_{\text{beam}} \overrightarrow{AM^*}\, m\, ds \equiv \int_{s=0}^{L} [(s+u)\hat{x}+v\hat{y}]m\, ds \triangleq (x_c\hat{x}) \int_{s=0}^{L} m\, ds. \tag{1}$$

Since C is also the center of mass for the undeformed beam, the above equation is also satisfied for $u = v = 0$. Therefore, eqn (1) yields the following relations for the elastic displacements $u(s,t)$ and $v(s,t)$,

$$\int_0^L u(s,t)m(s)\, ds = 0; \quad \int_0^L v(s,t)m(s)\, ds = 0. \tag{2a, b}$$

Since the reference line AB (which is rotating relative to the orbital reference line) is a principal axis of the deformed beam, the product of inertia $I_{xy} = \int_0^L (s-x_c+u)vm\, ds$ is zero. Making use of eqn (2b), the condition $I_{xy} = 0$ then reduces to

$$\int_0^L (s+u)vm\, ds = 0. \tag{3}$$

Equations (2a, b) and (3) will be needed later in this paper.

The beam is modeled as an Euler–Bernoulli beam with very large axial stiffness EA so that it is approximated as an inextensional beam (Crespo da Silva, 1988; 1991). Therefore, the quantities $u(s,t)$ and $v(s,t)$ are related by the following constraint equation due to inextensionality

$$(1+u')^2+v'^2 = 1 \tag{4}$$

where primes denote partial differentiation with respect to s.

In the next section, the differential equations of motion are generated via Hamilton's principle. For this, the expressions for the kinetic energy, and for the virtual work done by the forces applied to the beam, are needed. Letting dots denote partial differentiation with respect to time t, the absolute velocity of an arbitrary point M^* on the beam's reference line is (see Fig. 1)

$$\begin{aligned}\vec{v}_{M^*} = [R_c\hat{x}_1+(s-x_c+u)\hat{x}+v\hat{y}]^{\cdot} &= [\dot{u}-(\dot{\phi}+\dot{\theta})v+\dot{R}_c\cos\theta+\dot{\phi}R_c\sin\theta]\hat{x} \\ &\quad +[\dot{v}+(\dot{\phi}+\dot{\theta})(s-x_c+u)-\dot{R}_c\sin\theta+\dot{\phi}R_c\cos\theta]\hat{y}.\end{aligned} \tag{5}$$

If the (ξ, η, ζ) axes in Fig. 1 are chosen to be the principal axes of inertia of the cross-section at location s, and centered at the cross-section's center of mass, the specific kinetic energy, T, becomes (Crespo da Silva, 1991)

$$T = \tfrac{1}{2}m\vec{v}_{M^*}\cdot\vec{v}_{M^*} + \tfrac{1}{2}j_\zeta(\dot{\theta}_z+\dot{\theta}+\dot{\phi})^2 = \tfrac{1}{2}j_\zeta(\dot{\theta}_z+\dot{\theta}+\dot{\phi})^2$$
$$+\frac{m}{2}\{[\dot{u}-(\dot{\phi}+\dot{\theta})v]^2+[\dot{v}+(\dot{\phi}+\dot{\theta})(s-x_c+u)]^2+\dot{R}_c^2+\dot{\phi}^2R_c^2$$
$$+2[\dot{u}-(\dot{\phi}+\dot{\theta})v][\dot{R}_c\cos\theta+\dot{\phi}R_c\sin\theta]$$
$$+2[\dot{v}+(\dot{\phi}+\dot{\theta})(s-x_c+u)][\dot{\phi}R_c\cos\theta-\dot{R}_c\sin\theta]\} \quad (6)$$

where $\theta_z = \arctan[v'/(1+u')]$, and j_ζ is the specific mass moment of inertia of the beam's cross-section about its ζ axis.

If the beam material is Hookean and isotropic, and if the small effect of shear is neglected, the specific virtual work due to deformation is equal to $-D_\zeta\theta'_z\delta\theta'_z$, where D_ζ is the beam's bending stiffness about the ζ axis. For long and slender beams, the j_ζ terms, which are of the order of magnitude of the shear effects that have been neglected, have a negligible effect on the motion of the beam. Therefore, their effect will also be neglected from now on.

The specific virtual work due to the gravitational attraction from the central attracting point E, which we will denote as $(\delta W)_g$, is simply

$$(\delta W)_g = -\frac{Gm_E}{r_{M^*}^2}\delta r_{M^*} \triangleq Q_{u_g}\delta u+Q_{v_g}\delta v+Q_{\theta_g}\delta\theta+Q_{R_{cg}}\delta R_c \quad (7)$$

where G is the universal gravitational constant, and $r_{M^*} = |\overrightarrow{EM^*}|$ is the distance from E to M^*. By making use of the following expression for δr_{M^*} [see Fig. 1 and eqn (5)],

$$\delta r_{M^*} \equiv (\delta r_{M^*}^2)/(2r_{M^*})$$
$$= \{\delta[R_c^2+(s-x_c+u)^2+v^2+2R_c\hat{x}_1\cdot\overrightarrow{CM^*}]\}/(2r_{M^*}) \quad (8)$$

and by noticing that $\overrightarrow{CM^*} = (s-x_c+u)(\hat{x}_1\cos\theta+\hat{y}_1\sin\theta)+v(\hat{y}_1\cos\theta-\hat{x}_1\sin\theta)$, the specific generalized gravitational forces Q_{u_g}, Q_{v_g}, Q_{θ_g} and $Q_{R_{cg}}$ are then obtained as

$$Q_{u_g} = -\frac{Gm_Em}{R_c^3}(s-x_c+u+R_c\cos\theta)(r_{M^*}/R_c)^{-3} \quad (9a)$$

$$Q_{v_g} = \frac{Gm_Em}{R_c^3}(R_c\sin\theta-v)(r_{M^*}/R_c)^{-3} \quad (9b)$$

$$Q_{\theta_g} = \frac{Gm_Em}{R_c^3}R_c[(s+u-x_c)\sin\theta+v\cos\theta](r_{M^*}/R_c)^{-3} \quad (9c)$$

$$Q_{R_{cg}} = -\frac{Gm_Em}{R_c^3}[R_c+(s+u-x_c)\cos\theta-v\sin\theta](r_{M^*}/R_c)^{-3} \quad (9d)$$

where

$$(r_{M^*}/R_c)^{-3} = \left\{1+\frac{(s-x_c+u)^2+v^2}{R_c^2}+\frac{2}{R_c}[(s-x_c+u)\cos\theta-v\sin\theta]\right\}^{-3/2}$$
$$\approx 1-3\frac{(s-x_c+u)\cos\theta-v\sin\theta}{R_c}. \quad (10)$$

Expressing all other distributed forces that may be acting on the beam as

$$\vec{F} = F_\eta\hat{\eta}+F_\xi\hat{\xi} = (F_\xi\cos\theta_z-F_\eta\sin\theta_z)\hat{x}+(F_\xi\sin\theta_z+F_\eta\cos\theta_z)\hat{y}$$
$$= [(1+u')F_\xi-v'F_\eta]\hat{x}+[v'F_\xi+(1+u')F_\eta]\hat{y} \quad (11)$$

and modeling the virtual work due to structural damping as $-c\dot{v}\delta v$, the specific virtual work done by the nongravitational forces, which we will denote as δW_0, is then obtained as given below. The generalized forces $Q^*_{R_c}$ and Q^*_ϕ were added in eqn (12) in order to make the center of mass C travel on any trajectory that one may specify. Since damping due to the space environment (i.e. associated with $\delta\theta$, $\delta\phi$ and δR_c) is expected to be very small, it is neglected here for simplicity.

$$\begin{aligned}\delta W_0 &= -c\dot{v}\delta v + \vec{F}\cdot\delta(R_c\hat{x}_1 + \overrightarrow{CM^*}) + Q^*_{R_c}\delta R_c + Q^*_\phi\delta\phi \\ &= [(1+u')F_\xi - v'F_\eta]\,\delta u + [v'F_\xi + (1+u')F_\eta - c\dot{v}]\,\delta v \\ &\quad + \{(s+u-x_c)[(1+u')F_\eta + v'F_\xi] + v[v'F_\eta - (1+u')F_\xi]\}\,\delta\theta \\ &\quad + \{(s+u-x_c)[(1+u')F_\eta + v'F_\xi] + v[v'F_\eta - (1+u')F_\xi] \\ &\quad + R_c[F_\xi \sin(\theta+\theta_z) + F_\eta\cos(\theta+\theta_z)] + Q^*_\phi\}\,\delta\phi \\ &\quad + [F_\xi\cos(\theta+\theta_z) - F_\eta\sin(\theta+\theta_z) + Q^*_{R_c}]\,\delta R_c \\ &\triangleq Q_{u_0}\delta u + (Q_{v_0} - c\dot{v})\,\delta v + Q_{\theta_0}\delta\theta + Q_\phi\delta\phi + Q_{R_{c0}}\delta R_c. \end{aligned} \tag{12}$$

With δW_0 given by eqn (12), the total specific virtual work done by the distributed forces acting on the beam, δW, is then obtained as

$$\begin{aligned}\delta W = \delta W_g + \delta W_0 = -D_\zeta\theta_z'\delta\theta_z' &+ (Q_{u_g} + Q_{u_0})\,\delta u + (Q_{v_g} + Q_{v_0} - c\dot{v})\,\delta v \\ &+ (Q_{\theta_g} + Q_{\theta_0})\,\delta\theta + Q_\phi\delta\phi + (Q_{R_{cg}} + Q_{R_{c0}})\,\delta R_c. \end{aligned} \tag{13}$$

The expressions for T and δW are used in the next section to generate the differential equations of motion for the beam.

DIFFERENTIAL EQUATIONS OF MOTION

The differential equations of motion, and a boundary condition equation, can be readily obtained from Hamilton's extended principle, which is given as

$$\delta I = \int_{t=t_1}^{t_2}\int_{s=0}^{L}\{\delta(T) + \delta W + \tfrac{1}{2}\lambda[1-(1+u')^2 - v'^2]\}\,\mathrm{d}s\,\mathrm{d}t + \int_{t=t_1}^{t_2}\delta W_B\,\mathrm{d}t = 0 \tag{14}$$

where λ is a Lagrange multiplier that is used to handle the constraint, eqn (4), and δW_B is the virtual work associated with any force that may be applied at the boundaries, such as follower forces, which require an integration by parts in eqn (14) (Crespo da Silva and Glynn, 1978a). For simplicity, it will be assumed here that the beam is not subjected to such forces.

By taking the variation of the kinetic energy T, and by integrating by parts some of the terms in eqn (14), the following equations of motion and boundary condition equation are readily obtained after making use of eqns (2a, b) and (3), and after neglecting the j_ζ terms,

$$\left(\int_0^L m\,\mathrm{d}s\right)(\ddot{R}_c - \dot{\phi}^2 R_c) = -\frac{Gm_E}{R_c^2}\int_0^L m\,\mathrm{d}s + \int_0^L Q_{R_{c0}}\,\mathrm{d}s \tag{15}$$

$$\left(\int_0^L m\,\mathrm{d}s\right)(R_c^2\dot{\phi})^{\cdot} = \int_0^L Q_\phi\,\mathrm{d}s \tag{16}$$

$$\int_0^L m\left\{(s-x_c+u)^2\left(\ddot{\theta}+\ddot{\phi}+3\frac{Gm_E}{R_c^3}\sin\theta\cos\theta\right)+(s-x_c+u)\left[\ddot{v}+3\frac{Gm_E}{R_c^3}v\cos(2\theta)\right]\right.$$
$$\left.+2(\dot{\phi}+\dot{\theta})(s-x_c+u)\dot{u}-\ddot{u}v+[(\dot{\phi}+\dot{\theta})v^2]^{\cdot}-3\frac{Gm_E}{R_c^3}v^2\sin\theta\cos\theta\right\}\mathrm{d}s$$
$$=\int_0^L Q_{\theta_0}\,\mathrm{d}s \quad (17)$$

$$[(D_\zeta\theta_z')'v'+\lambda(1+u')]' \triangleq G_u' = m\left\{\ddot{u}-2(\dot{\phi}+\dot{\theta})\dot{v}-(\ddot{\phi}+\ddot{\theta})v\right.$$
$$\left.-(\dot{\phi}+\dot{\theta})^2(s-x_c+u)+(\ddot{R}_c-\dot{\phi}^2R_c)\cos\theta-\frac{1}{R_c}(R_c^2\dot{\phi})^{\cdot}\sin\theta\right\}-Q_{u_0}$$
$$+\frac{Gm_E}{R_c^3}m(s-x_c+u+R_c\cos\theta)\left[1-3\frac{(s-x_c+u)\cos\theta-v\sin\theta}{R_c}\right]$$
$$\approx m\left\{\ddot{u}-2(\dot{\phi}+\dot{\theta})\dot{v}-(\ddot{\phi}+\ddot{\theta})v-(2\dot{\phi}+\dot{\theta})(s-x_c+u)\dot{\theta}\right.$$
$$+\left(\frac{Gm_E}{R_c^3}-\dot{\phi}^2\right)(s-x_c+u)-3\frac{Gm_E}{R_c^3}[(s-x_c+u)\cos\theta-v\sin\theta]\cos\theta$$
$$\left.+\left[\ddot{R}_c+\left(\frac{Gm_E}{R_c^3}-\dot{\phi}^2\right)R_c\right]\cos\theta-\frac{1}{R_c}(R_c^2\dot{\phi})^{\cdot}\sin\theta\right\}-Q_{u_0} \quad (18)$$

$$[-(D_\zeta\theta_z')'(1+u')+\lambda v']' \triangleq G_v' = m\left\{\ddot{v}+2(\dot{\phi}+\dot{\theta})\dot{u}\right.$$
$$\left.+(\ddot{\phi}+\ddot{\theta})(s-x_c+u)-(\dot{\phi}+\dot{\theta})^2v-(\ddot{R}_c-\dot{\phi}^2R_c)\sin\theta+\frac{1}{R_c}(R_c^2\dot{\phi})^{\cdot}\cos\theta\right\}$$
$$-\frac{Gm_E}{R_c^3}m(R_c\sin\theta-v)\left[1-3\frac{(s-x_c+u)\cos\theta-v\sin\theta}{R_c}\right]+c\dot{v}-Q_{v_0}$$
$$\approx m\left\{\ddot{v}+2(\dot{\phi}+\dot{\theta})\dot{u}+(\ddot{\phi}+\ddot{\theta})(s-x_c+u)-(2\dot{\phi}+\dot{\theta})\dot{\theta}v\right.$$
$$+\left(\frac{Gm_E}{R_c^3}-\dot{\phi}^2\right)v-\left[\ddot{R}_c+\left(\frac{Gm_E}{R_c^3}-\dot{\phi}^2\right)R_c\right]\sin\theta+\frac{1}{R_c}(R_c^2\dot{\phi})^{\cdot}\cos\theta$$
$$\left.+3\frac{Gm_E}{R_c^3}[(s-x_c+u)\cos\theta-v\sin\theta]\sin\theta\right\}+c\dot{v}-Q_{v_0}. \quad (19)$$

In the above equations, very small terms that were inversely proportional to the distance R_c from E (the center of mass of the attracting body) to C, such as $v\sin\theta/R_c$, were neglected.

Considering the case where $\delta W_B = 0$, the following boundary condition equation is obtained from the terms that were integrated by parts in eqn (14)

$$\{G_u\delta u+G_v\delta v+D_\zeta\theta_z'\delta\theta_z\}_{s=0}^{L}=0. \quad (20)$$

For a free–free beam, the following boundary conditions are extracted from the above equation

$$G_u(s=0,t) = G_u(s=L,t) = 0 \tag{21a}$$

$$\theta_z'(0,t) = \theta_z'(L,t) = 0 \quad \therefore \quad v''(0,t) = v''(L,t) = 0 \tag{21b}$$

$$[(D_\zeta \theta_z')']_{s=0;s=L} = 0 \quad \therefore \quad v'''(0,t) = v'''(L,t) = 0. \tag{21c}$$

The elastic displacement $u(s,t)$ may be eliminated from the differential equations of motion. To do this, eqn (4) is first solved for $u'(s,t)$ and the result integrated from $s = L$ to $s = s$ to yield

$$u(s,t) = u(L,t) + \int_L^s [\sqrt{1-v'^2}-1]\,\mathrm{d}s. \tag{22}$$

An expression for $u(L,t)$ can be obtained by making use of the following identity

$$\int_0^L um\,\mathrm{d}s \equiv \left[u\int_0^s m\,\mathrm{d}s\right]_{s=0}^L - \int_0^L u'\left[\int_0^s m\,\mathrm{d}s\right]\mathrm{d}s \tag{23}$$

and of the result given by eqn (2a). By solving eqn (23) for $u(L,t)$, the following expression for $u(s,t)$ for a free–free beam is then obtained

$$u(s,t) = \frac{1}{\int_0^L m\,\mathrm{d}s}\int_0^L [\sqrt{1-v'^2}-1]\left[\int_0^s m\,\mathrm{d}s\right]\mathrm{d}s + \int_L^s [\sqrt{1-v'^2}-1]\,\mathrm{d}s. \tag{24}$$

Equations (15)–(19) are the full-nonlinear differential equations that govern the motion of a free–free beam subjected to the gravity gradient forces due to the central attracting body and to external distributed forces.

Equation (19) may be reduced to an integro-partial differential equation that does not involve the elastic displacement $u(s,t)$ and the Lagrange multiplier $\lambda(s,t)$. To do this, both sides of eqn (18) are integrated from $s = L$ to $s = s$, and the resulting equation is solved for the Lagrange multiplier λ. The solution for λ is then substituted into eqn (19) to obtain an alternate expression for G_v. With $u(s,t)$ given by eqn (24), the new expression for G_v to be used in eqn (19), obtained as indicated above, is

$$\begin{aligned} G_v = \frac{1}{\sqrt{1-v'^2}}\Bigg\{ &-(D_\zeta\theta_z')' + v'\int_L^s m\Bigg[\ddot{u} - 2(\dot\phi+\dot\theta)\dot{v} - (\ddot\phi+\ddot\theta)v - (2\dot\phi+\dot\theta)(s-x_c+u)\dot\theta \\ &+ \left(\frac{Gm_E}{R_c^3} - \dot\phi^2\right)(s-x_c+u) - 3\frac{Gm_E}{R_c^3}[(s-x_c+u)\cos\theta - v\sin\theta]\cos\theta \\ &+ \left[\ddot{R}_c + \left(\frac{Gm_E}{R_c^3} - \dot\phi^2\right)R_c\right]\cos\theta - \frac{1}{R_c}(R_c^2\dot\phi)^{\cdot}\sin\theta\Bigg]\mathrm{d}s - v'\int_L^s Q_{u_0}\,\mathrm{d}s\Bigg\}. \end{aligned} \tag{25}$$

EXPANDED EQUATIONS FOR A BEAM IN CIRCULAR ORBIT

To generate as much information as possible about the dynamic behavior of the beam, an analytical investigation of the motion is governed by eqns (15)–(17), and (19) with G_v given by eqn (25), is desirable. To do this, however, it becomes necessary that we restrict the type of motions to be investigated so that those equations become suitable for such analysis. The first step in doing this consists of determining the equilibrium solutions to those equations. For "small" perturbed motions about the equilibrium solutions, one can expand the nonlinear differential equations of motion in Taylor series about their equilibrium solution and retain the resulting polynomial nonlinearities to a desired degree in the perturbations. The resulting equations that are generated in this manner are then suitable for a perturbation analysis of the nonlinear motion. Such analysis reveals a wealth of information about the behavior of the system. Such equations are developed here for the case when both $v(s, t)$ and $\theta(t)$ are small, and when the beam is in a circular orbit so that R_c = constant and $\dot{\phi} = Gm_E/R_c^3 \triangleq \Omega_c$, which is also a constant. A study of the influence of satellite flexibility on the orbital motion has been presented by Misra and Modi (1978). Inspection of eqns (17) and (19) immediately discloses that $v = 0$, and $\theta = 0$ or $\theta = \pi$, are equilibrium solutions to those equations when $Q_{v_0} = 0$ and $Q_{\theta_0} = 0$. Here, eqns (17) and (19) are expanded about the equilibrium state $v = \theta = 0$, which is the stable equilibrium of the system.

To generate the perturbed equations for small v and θ, let ε be a small bookkeeping parameter that is introduced only to keep track of orders of magnitude. With $v = O(\varepsilon)$ and $\theta = O(\varepsilon)$, the following expansions are obtained for $u(s, t)$, which is given by eqn (24), and for $\theta_z(s, t)$:

$$u(s,t) = -\frac{1}{2\int_0^L m\,\mathrm{d}s}\int_0^L v'^2\left[\int_0^s m\,\mathrm{d}s\right]\mathrm{d}s - \frac{1}{2}\int_L^s v'^2\,\mathrm{d}s + O(\varepsilon^4) \tag{26a}$$

$$\theta_z = \arcsin v' = v'\left(1+\frac{v'^2}{6}\right) + O(\varepsilon^4). \tag{26b}$$

With G_v given by eqn (25), the $O(\varepsilon^3)$ expanded form of eqn (19) is then obtained as

$$\begin{aligned} &m\{\ddot{v}+\ddot{\theta}(s+u-x_c)+2(\Omega_c+\dot{\theta})\dot{u}-(2\Omega_c+\dot{\theta})\dot{\theta}v+3\Omega_c^2[(s-x_c)(\theta-\tfrac{2}{3}\theta^3)+u\theta-v\theta^2]\} \\ &\quad +(D_\zeta v'')''+c\dot{v}+\left\{v'(1+\tfrac{1}{2}v'^2)\int_L^s Q_{u_0}\,\mathrm{d}s+v'(D_\zeta v'v'')'+\tfrac{3}{2}\Omega_c^2 v'^3\int_L^s m(s-x_c)\,\mathrm{d}s\right\}' \\ &\quad -\left\{v'\int_L^s m[\ddot{u}-2(\Omega_c+\dot{\theta})\dot{v}-\ddot{\theta}v-(\dot{\theta}+2\Omega_c)(s-x_c)\dot{\theta}\right. \\ &\quad \left.-3\Omega_c^2(s+u-x_c-\theta^2(s-x_c)-v\theta)]\,\mathrm{d}s\right\}' = Q_{v_0} \end{aligned} \tag{27}$$

where $u(s, t)$ is given by eqn (26a). By proceeding in a similar manner, the $O(\varepsilon^3)$ expanded form of eqn (17) is obtained as given below:

$$\begin{aligned} &\int_0^L m\{(s-x_c)^2(\ddot{\theta}+3\Omega_c^2\theta-2\Omega_c^2\theta^3)+2(s-x_c)u(\ddot{\theta}+3\Omega_c^2\theta) \\ &\qquad +(s-x_c)(\ddot{v}+3\Omega_c^2 v-6\Omega_c^2 v\theta^2)+u(\ddot{v}+3\Omega_c^2 v) \\ &\qquad +2(\Omega_c+\dot{\theta})(s-x_c)\dot{u}-\ddot{u}v+[(\Omega_c+\dot{\theta})v^2]^{\boldsymbol{\cdot}}-3\Omega_c^2 v^2\theta\}\,\mathrm{d}s = \int_0^L Q_{\theta_0}\,\mathrm{d}s. \end{aligned} \tag{28}$$

Equations (27) and (28), with u given by eqn (26a), are the $O(\varepsilon^3)$ nonlinear equations that govern the coupled bending–pitching motion of a flexible beam in a circular orbit about a central attracting body such as the Earth. An approximate solution to these equations may be obtained by perturbation techniques such as those presented by Nayfeh and Mook (1989). To this end, a modal reduction technique may be applied to these equations to convert them into a set of nonlinear ordinary differential equations, which are then analysed in succession for the different levels of approximation.

NORMALIZATION AND MODAL REDUCTION OF THE EQUATIONS

To analyse the motion of the beam it is convenient to write the differential equations of motion in nondimensional form by introducing the following normalized quantities, indicated by a * superscript.

$$s^* = s/L; \quad v^* = v/L; \quad u^* = u/L; \quad x_c^* = x_c/L \tag{29a}$$

$$\mu(s^*) = m(s^*)\bigg/\int_0^1 m(s^*)\,\mathrm{d}s^*; \quad \beta_\zeta(s^*) = D_\zeta(s^*)\bigg/\int_0^1 D_\zeta(s^*)\,\mathrm{d}s^* \tag{29b}$$

$$t^* = t\sqrt{\frac{1}{L^4}\left[\int_0^1 D_\zeta(s^*)\,\mathrm{d}s^*\right]\bigg/\int_0^1 m(s^*)\,\mathrm{d}s^*} \tag{29c}$$

$$\omega_c^* = \Omega_c L^2\sqrt{\left[\int_0^1 m(s^*)\,\mathrm{d}s^*\right]\bigg/\left[\int_0^1 D_\zeta(s^*)\,\mathrm{d}s^*\right]} \tag{29d}$$

$$Q_{\alpha_0}^* = L^3 Q_{\alpha_0}\bigg/\int_0^1 D_\zeta(s^*)\,\mathrm{d}s^*\ (\alpha = u, v); \quad Q_\theta^* = L^2 Q_\theta\bigg/\int_0^1 D_\zeta(s^*)\,\mathrm{d}s^* \tag{29e}$$

$$c^* = cL^2\bigg/\sqrt{\left[\int_0^1 m(s^*)\,\mathrm{d}s^*\right]\left[\int_0^1 D_\zeta(s^*)\,\mathrm{d}s^*\right]}. \tag{29f}$$

The normalized equations are of the same form as eqns (27) and (28). To write them, one simply needs to replace the distributed mass m by μ and the stiffness D_ζ by β_ζ in those equations. For convenience in notation, the * superscript will be dropped from any normalized quantity referred to from this point forward.

To apply a modal reduction to the normalized equations, let us first look at the solution to their $O(\varepsilon)$ (i.e. linearized) counterpart in the absence of excitation and damping. In this case, the solution for the $O(\varepsilon)$ part of $v(s, t)$ is of the form

$$v(s,t) = \sum_{i=1}^{n} F_i(s) v_{t_i}(t).$$

Since eqn (2b) and the $O(\varepsilon)$ part of eqn (3) disclose that

$$\int_0^1 \mu(s) F_i(s)\,\mathrm{d}s = 0 \tag{30}$$

and that

$$\int_0^1 s\mu(s) F_i(s)\,\mathrm{d}s = 0 \tag{31}$$

it can be readily verified that, with $v_{t_i}(t) = A\cos(\omega_i t + B)$, and with Q_{u_0} being at least $O(\varepsilon)$, the eigenfunction $F_i(s)$ satisfies the following differential equation, which is obtained from eqns (27) and (28) with the boundary conditions for a free–free beam

$$(\beta_\zeta F_i'')'' - \mu\omega_i^2 F_i + 3\omega_c^2\left[F_i'\int_1^s (s - x_c)\mu\,\mathrm{d}s\right]' = 0. \tag{32}$$

Equation (32), with the boundary conditions $F_i''(0) = F_i'''(0) = F_i''(1) = F_i'''(1) = 0$, constitutes a two-point boundary value problem that can be solved numerically to determine the eigenfunction $F_i(s)$ and the frequency ω_i $(i = 1, 2, \ldots)$ for the general case when μ and β_ζ are functions of s. It can be easily verified that the eigenfunctions are orthogonal in the sense that $\int_0^1 F_i(s)F_j(s)\,ds = 0$ for $j \neq i$. For convenience, we will normalize the eigenfunction $F_i(s)$ so that $\int_0^1 \mu F_i^2\,ds = 1$.

An approximate solution to the nonlinear eqns (27) and (28) may be obtained by letting

$$v(s,t) = \sum_{i=1}^{n} F_i(s)v_{t_i}(t)$$

in those equations, multiplying the resulting δv-equation (eqn 27) by $F_j(s)$, and then integrating the result from $s = 0$ to $s = 1$. This yields a set of nonlinear ordinary differential equations for the temporal parts of the response, $v_{t_i}(t)$. To do this, only a one mode approximation for $v(s,t)$ will be considered here for simplicity. Thus, dropping the i subscript, the $O(\varepsilon^3)$ approximation for $u(s,t)$ given by eqn (26a) becomes

$$u(s,t) = -\frac{1}{2}\left[\frac{1}{\int_0^1 \mu\,ds}\int_0^1 F'^2\left(\int_0^s \mu\,ds\right)ds - \int_s^1 F'^2\,ds\right]v_t^2(t) \triangleq K_2(s)v_t^2. \tag{33}$$

Similarly, since the external force components F_η and F_ξ are at least $O(\varepsilon)$, the $O(\varepsilon^3)$ approximation for the generalized forces Q_{u0}, Q_{v0} and $Q_{\theta 0}$, given by eqn (12), become

$$Q_{u0} = \left[1 - \frac{F'^2(s)}{2}v_t^2(t)\right]F_\xi(s,t) - F'(s)F_\eta(s,t)v_t(t) \tag{34a}$$

$$Q_{v0} = F'(s)F_\xi(s,t)v_t(t) + \left[1 - \frac{F'^2(s)}{2}v_t^2(t)\right]F_\eta(s,t) \tag{34b}$$

$$Q_{\theta 0} = (s - x_c)\left\{F'(s)F_\xi(s,t)v_t(t) + \left[1 - \frac{F'^2(s)}{2}v_t^2(t)\right]F_\eta(s,t)\right\}$$
$$+ K_2(s)F_\eta(s,t)v_t^2(t) - [F_\xi(s,t) - F'(s)F_\eta(s,t)v_t(t)]v_t(t). \tag{34c}$$

By introducing the following constant quantities which will appear as coefficients in the reduced differential equations,

$$\beta_1 = \int_0^1 F\left[F'\int_1^s (s-x_c)\mu\,ds\right]'ds = -\int_0^1 F'^2\int_1^s (s-x_c)\mu\,ds\,ds \tag{35a}$$

$$\beta_2 = -\int_0^1 F\left[F'\int_1^s \mu K_2(s)\,ds\right]'ds = \int_0^1 F'^2\int_1^s \mu K_2(s)\,ds\,ds \tag{35b}$$

$$\beta_3 = \int_0^1 F\left\{F'(\beta_\xi F'F'')' + \tfrac{3}{2}\omega_c^2\left[F'^3\int_1^s (s-x_c)\mu\,ds\right]'\right\}ds - 3\omega_c^2\beta_2 \tag{35c}$$

and by noticing that

$$\int_0^1 \mu(s)K_2(s)F(s)\,ds = 0 \tag{36}$$

as required by the $O(\varepsilon^3)$ part of eqn (3), and that [see eqn (A1) in the Appendix]

$$\int_0^1 F\left[F'\int_1^s \mu F\,ds\right]'ds = 0 \tag{37}$$

the following reduced ordinary nonlinear differential equation is obtained from eqn (27):

$$\ddot{v}_t + c\dot{v}_t + \omega^2 v_t + (\beta_1 - 1)(2\omega_c + \dot{\theta})\dot{\theta} v_t - 3\omega_c^2(\beta_1 + 1)\theta^2 v_t + \beta_2 v_t (v_t^2)^{\cdot\cdot} + \beta_3 v_t^3$$
$$= \int_0^1 FF_\eta \, ds - v_t \int_0^1 FF'' \left[\int_1^s F_\xi \, ds \right] ds + v_t^2 \int_0^1 F \left[F'' \int_1^s F' F_\eta \, ds + \tfrac{1}{2} F'^2 F_\eta \right] ds. \quad (38)$$

It should be noted that eqn (32) was used to eliminate a term in $\int_0^1 F(\beta_\xi F'')''\, ds$ that appears in the intermediate equation that leads to eqn (38).

In addition, by introducing the following relationship that is derived in the Appendix [see eqn (A2) in the Appendix]

$$\int_0^1 (s - x_c)\mu K_2 \, ds = -\beta_1/2 \quad (39)$$

eqn (28) also yields the following reduced ordinary differential equation

$$\left[\int_0^1 (s - x_c)^2 \mu \, ds \right] (\ddot{\theta} + 3\omega_c^2 \theta - 2\omega_c^2 \theta^3) - \beta_1 [v_t^2(\ddot{\theta} + 3\omega_c^2 \theta) + (\omega_c + \dot{\theta})(v_t^2)^{\cdot}]$$
$$+ [(\omega_c + \dot{\theta}) v_t^2]^{\cdot} - 3\omega_c^2 v_t^2 \theta = \int_0^1 (s - x_c) F_\eta \, ds + v_t \int_0^1 [(s - x_c)F' - F] F_\xi \, ds$$
$$+ v_t^2 \int_0^1 [K_2 + FF' - \tfrac{1}{2}(s - x_c)F'^2] F_\eta \, ds. \quad (40)$$

Equations (38) and (40) are the reduced differential equations for $v_t(t)$ and $\theta(t)$ with polynomial nonlinearities up to cubic order in the perturbed variables. These differential equations include all the nonlinearities due to deformation of the beam and due to the gravity gradient effect. The nonlinearities due to deformation consist of inertia and curvature terms.

Before closing, it is instructive to look at typical values for the constant coefficients in the reduced equations and at the frequencies of oscillation, ω_i, for different values of the orbital parameter ω_c. Table 1 shows the numerical values of ω_i, β_1, β_2 and β_3 for several values of the normalized orbital angular speed ω_c for a beam with $m(s)$ = constant (i.e. $\mu = 1$). The values shown in Table 1 were obtained by integrating eqn (32) numerically using the transition-matrix technique presented in detail by Crespo da Silva *et al.* (1991). The determination of the eigenfunction $F_i(s)$ and of the associated eigenfrequency ω_i (and, thus, of the constants β_1, β_2 and β_3) for the more general case where $m = m(s)$ presents no major difficulties when the method presented in Crespo da Silva *et al.* (1991) is applied to eqn (32) even when that equation has variable coefficients.

Figure 2 shows a plot of the normalized natural frquency for the first mode, ω_1, versus the normalized orbital angular speed ω_c for $\mu = 1$. For small values of ω_c one obtains $\omega_1 \approx 4.73^2 \approx 22.37$, which is the value corresponding to $\omega_c = 0$ when the beam is not subjected to the gravity-gradient moment. Note that, as disclosed by eqn (32), the limiting case $\omega_c = 0$ corresponds to a free–free beam in a constant gravitational field (which is the classical case treated in structural mechanics books). For comparison purposes, the natural

Table 1. Values of ω_i, β_1, β_2 and β_3 for $m(s)$ = constant ($\mu = 1$)

	First mode			Second mode			Third mode		
	$\omega_c = 1$	$\omega_c = 5$	$\omega_c = 10$	$\omega_c = 1$	$\omega_c = 5$	$\omega_c = 10$	$\omega_c = 1$	$\omega_c = 5$	$\omega_c = 10$
ω_i	22.577	27.003	37.598	61.828	65.443	75.596	121.04	124.36	134.16
β_1	3.0496	3.0463	3.038	6.397	6.383	6.346	11.32	11.30	11.23
β_2	61.2	61.208	61.263	262.7	262.17	260.79	799.3	796.67	789.1
β_3	20,689	23,306	31,865	361,192	378,220	433,965	2,444,457	2,498,448	2,673,243

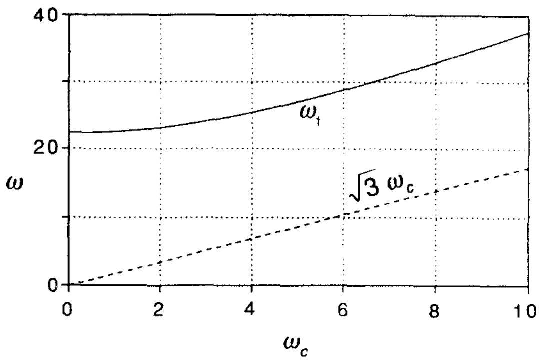

Fig. 2. Normalized natural frequencies for bending (ω_1), and pitch, for a beam in a circular orbit.

frequency of the θ-motion, $\omega_\theta = \sqrt{3}\omega_c$, is also shown in that figure. For an aluminum beam (AL2024-T4, $E = 73 \times 10^9$ N m^{-2}, $\rho = 2.77$ g cm^{-3}) of length L, with a hollow square cross-section of length b and thickness $\Delta \ll b$, one obtains for a low Earth orbit with period approximately equal to 86.4 minutes,

$$\omega_c = \Omega_c(L^2/b)\sqrt{(12\rho/E)[1+(1-2\Delta/b)^2]} \approx 6\times 10^{-7}L^2/b,$$

with L and b in meters. This gives, for example, $\omega_c \approx 0.02$ for a beam with $L = 100$ m and $b = 29$ cm. Thus, it is seen that the nondimensional orbital angular speed ω_c takes larger values as the length of the beam is increased. The larger values of ω_c illustrated in Table 1 corrrespond, at present, to unrealistically long beams (or, more precisely, very large values of L^2/b).

It is worth noting that eqns (38) and (40) disclose that the coupled v_t–θ motion exhibits internal resonances when ω_θ is near a bending natural frequency ω_i, or when ω_θ is near $2\omega_i$. However, since $\omega_i > \omega_\theta$ [i.e. any natural frequency ω_i ($i = 1, 2, \ldots$) is always greater than the pitch natural frequency $\omega_\theta = \sqrt{3}\omega_c$], as indicated by the results shown in Fig. 2, internal resonances that require $\omega_i \leqslant \omega_\theta$ are not physically possible.

SUMMARY AND CONCLUSIONS

The mathematically consistent nonlinear differential equations governing the coupled flexure–pitch motion for a beam in orbit were formulated in this paper. The formulation used here, which is based on the work presented by Crespo da Silva and Glynn (1978a, b) and by Crespo da Silva (1988, 1991) dealing with nonlinear dynamics of beams, accounts for all geometric nonlinearities in the system, and for nonlinearities due to orbital effects. The full nonlinear equations were expanded to include all the nonlinearities up to cubic order in a bookkeeping parameter ε. The beam material was assumed to be linear and, thus, the nonlinearities due to deformation are caused by changes in the geometry of the system. These include inertia nonlinearities, and nonlinear terms arising from the expression for the beam's curvature. The equations also contain second and third degree—i.e. $O(\varepsilon^2)$ and $O(\varepsilon^3)$—nonlinear coupling terms between the pitch and bending motions of the beam. Some of the nonlinear terms in the equations of motion are multiplied by the "Galerkin coefficients" β_1, β_2 and β_3. It should be noted that if all nonlinear terms in $v_t(t)$, as well as the β_1 terms, are neglected in eqn (38), the resulting equation yields eqn (4) in Kumar and Bainum (1980). Equation (3) in Kumar and Bainum (1980) corresponds to eqn (40) in this paper if all the nonlinear terms involving v_t are disregarded in the latter equation. Many, but not all, of the nonlinear terms that appear in eqn (27) were also found in the work reported by Ashley (1967). The missing terms in eqn (49) in Ashley (1967) involve the elastic deformation $u(t)$ and also terms arising from nonlinearities in the expression for the curvature of the beam. It is also noted that eqn (48) in Ashley (1967) does not contain all the nonlinear terms shown in eqn (28) developed in this paper. However, it is interesting to note that the terms that are missing in eqn (48) in Ashley (1967) do not contribute to the reduced equation, eqn (40), after Galerkin's procedure is applied. The analysis of the coupled pitch-bending motion of the beam, based on eqns (38) and (40) developed here, is presented in Part II of this work (Crespo da Silva and Zaretzky, 1993).

REFERENCES

Ashley, H. (1967). Observations on the dynamic behavior of large flexible bodies in orbit. *AIAA Jl* **5**, 460–469.

Bainum, P. M. and Kumar, V. K. (1982). Dynamics of orbiting flexible beams and platforms in the horizontal orientation. *Acta Astronautica* **9**, 119–127.

Budynas, R. and Poli, C. (1971). Planar motion of a large flexible satellite. *AIAA Jl* **9**, 2099–2102.

Crespo da Silva, M. R. M. (1991). Equations for nonlinear analysis of 3D motions of beams. *Appl. Mech. Rev.* **44**(11), Part 2, 51–59.

Crespo da Silva, M. R. M. (1988). Nonlinear flexural–flexural–torsional–extensional dynamics of beams—I. Formulation. *Int. J. Solids Structures* **24**, 1225–1234.

Crespo da Silva, M. R. M. and Glynn, C. C. (1978a). Nonlinear flexural–flexural–torsional dynamics of inextensional beams—I. Equations of motion. *J. Struct. Mech.* **6**, 437–448.

Crespo da Silva, M. R. M. and Glynn, C. C. (1978b). Nonlinear flexural–flexural–torsional dynamics of inextensional beams—II. Forced motions. *J. Struct. Mech.* **6**, 449–461.

Crespo da Silva, M. R. M. and Zaretzky, C. L. (1993). Nonlinear dynamics of a flexible beam in a central gravitational field—II. Nonlinear motions in circular orbit. *Int. J. Solids Structures* **30**, 2301–2316.

Crespo da Silva, M. R. M., Zaretzky, C. L. and Hodges, D. H. (1991). Effects of approximations on the static and dynamic response of a cantilever with a tip mass. *Int. J. Solids Structures* **27**, 565–583.

Etkin, B. and Hughes, P. C. (1967). Explanation of the anomalous spin behavior of satellites with long, flexible antennae. *J. Spacecraft Rockets* **4**, 1139–1145.

Hughes, P. C. (1986). *Spacecraft Attitude Dynamics*. Wiley, New York.

Krishna, R. and Bainum, P. M. (1984). Effect of solar radiation disturbance on a flexible beam in orbit. *AIAA Jl* **22**, 677–682.

Kumar, V. K. and Bainum, P. M. (1980). Dynamics of a flexible body in orbit. *J. Guidance Control* **3**, 90–92.

Misra, A. K. and Modi, V. J. (1978). The influence of satellite flexibility on orbital motion. *Celestial Mech.* **17**, 145–165.

Modi, V. J. (1974). Attitude dynamics of satellites with flexible appendages—a brief review. *J. Spacecraft Rockets* **11**, 743–751.

Modi, V. J. and Ng, A. C. (1989). Dynamics of axisymmetric satellites under the influence of solar radiation. *J. Astronautical Sci.* **37**, 17–41.

Nayfeh, A. H. and Mook, D. T. (1989). *Nonlinear oscillations*. Wiley Interscience, New York.

APPENDIX

The following results, obtained by several integration by parts, as shown below, were used when generating eqns (38) and (40).

$$\int_0^1 F\left[F'\int_1^s \mu F\,ds\right]' ds = \underbrace{\left[FF'\int_1^s \mu F\,ds\right]_{s=0}^1}_{=0} - \int_0^1 F'^2\int_1^s \mu F\,ds\,ds$$

$$= \underbrace{\left[\int_1^s \mu F\,ds\int_1^s F'^2\,ds\right]_{s=0}^1}_{=0} + \int_0^1 \mu F\left(\int_1^s F'^2\,ds\right)ds$$

$$= \int_0^1 \mu F\left\{-2K_2 - \frac{1}{\int_0^1 \mu\,ds}\int_0^1 F'^2\left(\int_0^s \mu\,ds\right)ds\right\}ds = 0 \tag{A1}$$

$$\beta_1 = -\int_0^1 F'^2\int_1^s (s-x_c)\mu\,ds\,ds$$

$$= -\underbrace{\left[\left(\int_1^s (s-x_c)\mu\,ds\right)\int_1^s F'^2\,ds\right]_{s=0}^1}_{=0} + \int_0^1 (s-x_c)\mu\left(\int_1^s F'^2\,ds\right)ds$$

$$= \int_0^1 (s-x_c)\mu\left\{-2K_2 - \frac{1}{\int_0^1 \mu\,ds}\int_0^1 F'^2\left(\int_0^s \mu\,ds\right)ds\right\}ds = -2\int_0^1 (s-x_c)\mu K_2\,ds. \tag{A2}$$